

Deformation Quantization of Nonholonomic Almost Kähler Models and Einstein Gravity

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Abstract

Nonholonomic distributions and adapted frame structures on (pseudo) Riemannian manifolds of even dimension are employed to build structures equivalent to almost Kähler geometry and which allows to perform a Fedosov-like quantization of gravity. The nonlinear connection formalism that was formally elaborated for Lagrange and Finsler geometry is implemented in classical and quantum Einstein gravity.

Keywords: Deformation quantization, quantum gravity, Einstein spaces, Finsler and Lagrange geometry, almost Kähler geometry.

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1 Introduction

In recent years a set of important results in quantum gravity originated from Loop Quantum Gravity (for reviews and discussion of results, including previous canonical, topological, perturbative and other approaches, see Refs. [1, 2]) that were developed in many cases as an alternative, see discussion in [3], to stringy models of gravity (as general references, see [4, 5, 6]).

Following different geometrical and nonlinear functional analytical methods, a number of fundamental results were obtained based on deformation quantization [7, 8, 9, 10, 11]. Some early attempts to apply the ideas and results from deformation quantization to gravitational fields can be found,

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for instance, in commutative and noncommutative self-dual gravity [12, 13], W -gravity [14] and (recently) in linearized gravity [15]. A geometric quantization formalism for Einstein's theory of gravity and its gauge generalizations has not been elaborated yet. One of the main differences between the former canonical, perturbative and loop geometry approaches to quantum gravity and deformation quantization consists in the fact that geometric quantization methods were elaborated in curved spacetimes endowed with a certain symplectic, Poisson, or almost Kähler structure, ...etc on the (co) tangent bundles, or, for instance, for Lie algebroids. It was believed that symplectic structures and generalizations cannot be introduced in a canonical (unique) form in real Einstein manifolds.

In our recent work [16, 17, 18], we proved that almost Kähler geometries can be obtained canonically for (pseudo) Riemannian manifolds of even dimensions, including in Einstein gravity, if the spacetime manifold is equipped with a frame structure and the associated nonlinear connection (N-connection). Briefly, a N-connection is defined in terms of a nonholonomic ¹ distribution splitting the tangent space of the spacetime manifold into conventional horizontal and vertical subspaces. This means that a frame structure with mixed holonomic and nonholonomic variables is being prescribed in the spacetime manifold.² The N-connection components are induced by certain off-diagonal elements of a metric ³ which allows us to construct a canonical almost Kähler geometry by deforming nonholonomically the fundamental geometric objects of the (pseudo) Riemannian geometry.⁴

The goal of this paper is to provide a natural Fedosov quantization procedure of four dimensional (pseudo) Riemannian and Einstein manifolds following the N-connection formalism formally elaborated in Finsler and Lagrange geometry [20, 21] and generalized to nonholonomic manifolds and quantum Lagrange-Finsler spaces in our partner work [16, 17, 18, 19] (readers are recommended to see them in advance).⁵ We shall develop our con-

¹In the mathematical and physics literature different terminology is used for this geometric property, like nonintegrable, non-holonomic and anholonomic — all them are equivalent

²One can use any system of reference and coordinates but certain constructions are naturally adapted to some prescribed nonholonomic structures

³in general, a four dimensional spacetime metric cannot be diagonalized by coordinate transformations

⁴it should be noted that physicists use the terms pseudo-Euclidean and pseudo-Riemannian geometry but mathematicians are familiar, for instance, with the term semi-Riemannian

⁵ An integral and tensor calculus on manifolds provided with an N-connection struc-

struction for metric compatible connections with an effective torsion and a Neijenhuis structure induced nonholonomically by the off-diagonal metric components.

The paper is organized as follows: In section 2 we show how (pseudo) Riemannian spaces equipped with prescribed nonholonomic distributions and nonlinear connection structures (i.e. N-anholonomic manifolds) can be modeled equivalently in terms of almost Kähler geometries. Section 3 is devoted to Fedosov's quantization of such N-anholonomic spaces. In section 4 we briefly conclude that the nonholonomic frame method, when applied to the deformation quantization procedure, provides a natural geometric quantization method to any solution of the field equations in Einstein gravity.

2 Almost Kähler Models of Nonholnomic (pseudo) Riemannian Spaces

The aim of this section is to show how by prescribing corresponding distributions (equivalently, some classes of nonholonomic frame structures) we can model a (pseudo) Riemannian manifold as an almost Kähler space.

We consider a (pseudo) Riemann manifold V^{2n} , $\dim V^{2n} = 2n$, where $n \geq 2$, of necessary smooth class.⁶ The local coordinates on V^{2n} are labelled in the form $u^\alpha = (x^i, y^a)$, or $u = (x, y)$, where indices run through the values $i, j, \dots = 1, 2, \dots, n$ and $a, b, \dots = n+1, n+2, \dots, n+n$, where x^i and y^a are called respectively the horizontal c (h) and vertical (v) coordinates.

A nonlinear connection (N-connection) \mathbf{N} on V^{2n} is defined as a Whitney sum (nonholonomic distribution) on the tangent bundle TV^{2n} ,

$$TV^{2n} = h(V^{2n}) \oplus v(V^{2n}), \quad (1)$$

with a global splitting into conventional h- and v-subspaces, given locally by a set of coefficients $N_i^a(x, y)$.⁷ The curvature of a N-connection is (by

ture requires a more sophisticate system of notations, see details in the above-mentioned partner works. We shall use the Einstein summation convention for local expressions.

⁶For the Einstein gravity theory, $2n = 4$. In this paper, for simplicity, we shall elaborate the geometric constructions starting with smooth classical spacetime manifolds. Nevertheless, we have to keep in mind that certain physical situations may request some more special geometric constructions when some functions and/or their derivatives are not smooth in some points/regions (for instance, black hole solutions). For such cases, we have to "relax" the general smooth manifold condition and postulate that we work with spacetimes and geometric objects of "necessary" smooth class.

⁷defined with respect to a coordinate basis $\partial_\alpha = \partial/\partial u^\alpha = (\partial_i = \partial/\partial x^i, \partial_a = \partial/\partial y^a)$ and its dual $du^\beta = (dx^j, dy^b)$; here we note that the subclass of linear connections consists of a particular case when $N_i^a = \Gamma_{bi}^a(x)y^b$

definition) just the Neijenhuis tensor

$$\Omega_{ij}^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}.$$

In this work, the spacetimes will be modelled as N-anholonomic manifolds \mathbf{V}^{2n} , i.e. (pseudo) Riemannian manifolds with prescribed nonholonomic distributions defining N-connection structures. For $n = 2$, fixing the Minkowski signature $(+---)$, we get a $2+2$ decomposition and can develop a respective nonholonomic splitting formalism alternatively to the so-called ADM (Arnowit, Deser and Misner) $(3+1)$ -decomposition (see, for instance, Ref. [22]). We are going to elaborate a canonical almost symplectic formalism which follows naturally from certain type $(2+2)$ -decompositions.

Having prescribed on \mathbf{V}^{2n} a N-connection structure $\mathbf{N} = \{N_j^a\}$, we can define a frame structure with coefficients depending linearly on N_j^a , denoted $\mathbf{e}_\nu = (\mathbf{e}_i, e_a)$, where

$$\mathbf{e}_i = \frac{\partial}{\partial x^i} - N_i^a(u) \frac{\partial}{\partial y^a} \text{ and } e_a = \frac{\partial}{\partial y^a}, \quad (2)$$

and the dual frame (coframe) structure is $\mathbf{e}^\mu = (e^i, \mathbf{e}^a)$, where

$$e^i = dx^i \text{ and } \mathbf{e}^a = dy^a + N_i^a(u) dx^i, \quad (3)$$

satisfying nontrivial nonholonomy relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma \quad (4)$$

with (antisymmetric) anholonomy coefficients $W_{ia}^b = \partial_a N_i^b$ and $W_{ji}^a = \Omega_{ij}^a$.⁸

Any metric $\mathbf{g} = \{g_{\alpha\beta}\}$ on \mathbf{V}^{2n} , with symmetric coefficients $g_{\alpha\beta}$ defined with respect to a coordinate dual basis $du^\alpha = (dx^i, dy^a)$, can be represented equivalently as d-tensor fields

$$\mathbf{g} = g_{ij}(x, y) e^i \otimes e^j + h_{ab}(x, y) \mathbf{e}^a \otimes \mathbf{e}^b, \quad (5)$$

$$= g_{ij}(x, y) e^i \otimes e^j + g_{a'b'}(x, y) \check{\mathbf{e}}^{a'} \otimes \check{\mathbf{e}}^{b'}, \quad (6)$$

where

$$h_{ab}(u) = g_{a'b'}(u) \mathbf{e}_a^{a'}(u) \mathbf{e}_b^{b'}(u) \text{ and } \check{\mathbf{e}}^{a'} = \mathbf{e}_a^{a'}(u) \mathbf{e}^a \quad (7)$$

⁸We use boldface symbols for the spaces (geometric objects) provided with N-connection structure (adapted to the h- and v-splitting defined by (1), we call them to be N-adapted). The N-adapted tensors, vectors, forms etc are called respectively distinguished tensors, vectors etc, in brief, d-tensors, d-vectors, d-forms etc.

are the v-components of the "new" N-adapted co-bases $\check{\mathbf{e}}^\mu = (e^i, \check{\mathbf{e}}^{a'})$, being dual to $\check{\mathbf{e}}_\nu = (\check{\mathbf{e}}_i, e_{a'})$.⁹ With respect to "v" bases, the respective coefficients g_{ij} are equal to $g_{a'b'}$. In a particular case, we can consider that a metric $\mathbf{g} = \{g_{\alpha\beta}\}$ is a solution of the Einstein equations on \mathbf{V}^{2n} .

From the class of affine connections on \mathbf{V}^{2n} , one prefers to work with N-adapted linear connections, called distinguished connections (in brief, d-connections). A d-connection $\mathbf{D} = (hD; vD) = \{\Gamma_{\beta\gamma}^\alpha = (L_{jk}^i, L_{bk}^a; C_{jc}^i, C_{bc}^a)\}$, with local coefficients computed with respect to (2) and (3), preserves under parallel transports the distribution (1). For a metric compatible d-connection \mathbf{D} , $\mathbf{D}_X \mathbf{g} = 0$, for any d-vector \mathbf{X} .

We can define respectively the torsion and curvature tensors,

$$\mathbf{T}(\mathbf{X}, \mathbf{Y}) \doteq \mathbf{D}_X \mathbf{Y} - \mathbf{D}_Y \mathbf{X} - [\mathbf{X}, \mathbf{Y}], \quad (8)$$

$$\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} \doteq \mathbf{D}_X \mathbf{D}_Y \mathbf{Z} - \mathbf{D}_Y \mathbf{D}_X \mathbf{Z} - \mathbf{D}_{[\mathbf{X}, \mathbf{Y}]} \mathbf{Z}, \quad (9)$$

where the symbol " \doteq " states "by definition" and $[\mathbf{X}, \mathbf{Y}] \doteq \mathbf{X}\mathbf{Y} - \mathbf{Y}\mathbf{X}$, for any d-vectors \mathbf{X} and \mathbf{Y} . With respect to fixed local bases \mathbf{e}_α and \mathbf{e}^β , the coefficients $\mathbf{T} = \{\mathbf{T}_{\beta\gamma}^\alpha\}$ and $\mathbf{R} = \{\mathbf{R}_{\beta\gamma\tau}^\alpha\}$ can be computed by introducing $\mathbf{X} \rightarrow \mathbf{e}_\alpha$, $\mathbf{Y} \rightarrow \mathbf{e}_\beta$, $\mathbf{Z} \rightarrow \mathbf{e}_\gamma$ into respective formulas (8) and (9).

One uses three important geometric objects: the Ricci tensor, $Ric(\mathbf{D}) = \{\mathbf{R}_{\beta\gamma} \doteq \mathbf{R}_{\beta\gamma\alpha}^\alpha\}$, the scalar curvature, ${}^s R \doteq \mathbf{g}^{\alpha\beta} \mathbf{R}_{\alpha\beta}$ ($\mathbf{g}^{\alpha\beta}$ being the inverse matrix to $\mathbf{g}_{\alpha\beta}$), and the Einstein tensor, $\mathbf{E} = \{\mathbf{E}_{\alpha\beta} \doteq \mathbf{R}_{\alpha\beta} - \frac{1}{2} \mathbf{g}_{\alpha\beta} {}^s R\}$. In Einstein gravity, one works with the Levi Civita connection $\nabla = \{\Gamma_{\beta\gamma}^\alpha\}$ uniquely defined by a metric \mathbf{g} in order to be metric compatible, $\nabla \mathbf{g} = 0$, and torsionless, ${}^t T_{\beta\gamma}^\alpha = 0$. We emphasize that ∇ is not a d-connection because it does not preserve under parallel transports the splitting (1) (that is why we do not use "boldface" letters).

For our purposes (to elaborate certain models of deformation quantization of gravity), it is convenient to work with two classes of metric compatible d-connections completely defined by a metric \mathbf{g} and for a $2 + 2$, in general, nonholonomic, splitting:

The first one is the so-called normal (in some cases, it is called also canonical) d-connection $\hat{\mathbf{D}} = (h\hat{D}; v\hat{D}) = \{\hat{\Gamma}_{\beta\gamma}^\alpha\}$, or $h\hat{D} = \{\hat{L}_{jk}^i\}$ and $v\hat{D} =$

⁹to get locally a diagonal Minkowski metric, we may consider (formally) that some local base vectors and corresponding coordinates are proportional to the complex imaginary unity " i " even in classical gravity we work with real (pseudo) Riemannian spaces

$\{\widehat{C}_{jc}^i\}$,¹⁰ when with respect to the N-adapted bases (2) and (3),

$$\widehat{L}_{jk}^i = \frac{1}{2}g^{ih}(\mathbf{e}_k g_{jh} + \mathbf{e}_j g_{kh} - \mathbf{e}_h g_{jk}), \quad \widehat{C}_{bc}^a = \frac{1}{2}h^{ae}(e_b h_{ec} + e_c h_{eb} - e_e h_{bc}). \quad (10)$$

This connection is uniquely defined by the metric structure (5) to satisfy the conditions $\widehat{\mathbf{D}}\mathbf{X}\mathbf{g} = 0$ and $\widehat{T}_{jk}^i = 0$ and $\widehat{T}_{bc}^a = 0$ (when torsion vanishes in the h- and v-subspaces); we note that, in general, the torsion $\widehat{\mathbf{T}}_{\beta\gamma}^\alpha$ has nontrivial torsion components

$$\widehat{T}_{jc}^i = \widehat{C}_{jc}^i, \widehat{T}_{ij}^a = \Omega_{ij}^a, \widehat{T}_{ib}^a = e_b N_i^a - \widehat{L}_{bi}^a, \quad (11)$$

which are induced by respective nonholonomic deformations and also completely defined by the N-connection and d-metric coefficients.

The second preferred metric compatible d-connection ${}^K\mathbf{D}$ is the same $\widehat{\mathbf{D}}$ but with the coefficients re-adapted with respect to the bases $\check{\mathbf{e}}_{\nu'} = (\delta_{\check{i}}^i \check{\mathbf{e}}_i, e_{a'})$ and $\check{\mathbf{e}}^{\mu'} = (e^{i'} = \delta_{\check{i}}^{i'} e^i, \check{\mathbf{e}}^{a'})$ (we use different symbols because we consider a different N-adapted frame with redefined local coefficients of N-connection, $\check{N}_{\check{i}}^{a'}(u) = \mathbf{e}_{\check{a}'}^{a'}(u) N_i^a(u)$, which defines a different nonholonomic distribution for fixed structures $\mathbf{e}_{\check{a}'}^{a'}(u)$ and $N_i^a(u)$). In general, we write $\check{\mathbf{e}}_{\nu'} = \check{\mathbf{e}}_{\nu'}^v(u) \mathbf{e}_\nu$ and $\check{\mathbf{e}}^{\nu'} = \check{\mathbf{e}}^{\nu'}_v(u) \mathbf{e}^v$, where the matrix $\check{\mathbf{e}}_{\check{a}'}^{a'}$ is inverse to $\check{\mathbf{e}}_{\check{a}'}^v$ defined correspondingly by $\delta_{\check{i}}^i, \delta_{\check{a}'}^a$, and transforms with nontrivial $\mathbf{e}_{\check{a}'}^{a'}$ stated by nonholonomic deformations (7). The N-adapted coefficients of ${}^K\mathbf{D}$ can be expressed through the N-adapted coefficients of $\widehat{\mathbf{D}}$ given by (10),

$${}^K\Gamma_{\beta'\gamma'}^{\alpha'} = \check{\mathbf{e}}_{\alpha'}^{\alpha'} \check{\mathbf{e}}_{\beta'}^{\beta} \check{\mathbf{e}}_{\gamma'}^{\gamma} \Gamma_{\beta\gamma}^\alpha + \check{\mathbf{e}}_{\alpha'}^{\alpha'} \mathbf{e}_\gamma(\check{\mathbf{e}}_{\beta'}^\alpha). \quad (12)$$

The torsion of this d-connection satisfies the condition

$${}^K\mathbf{T}_{\beta\gamma}^\alpha = (1/4){}^K\Omega_{\beta\gamma}^\alpha, \quad (13)$$

where ${}^K\Omega_{\beta\gamma}^\alpha$ are the N-adapted coefficients of the Nijenhuis tensor

$${}^K\Omega(\mathbf{X}, \mathbf{Y}) \doteq [\check{\mathbf{J}}\mathbf{X}, \check{\mathbf{J}}\mathbf{Y}] - \check{\mathbf{J}}[\check{\mathbf{J}}\mathbf{X}, \mathbf{Y}] - \check{\mathbf{J}}[\mathbf{X}, \mathbf{Y}] - [\mathbf{X}, \mathbf{Y}] \quad (14)$$

defined for the almost complex structure $\check{\mathbf{J}} : \check{\mathbf{J}}(\check{\mathbf{e}}_i) = -e_i$ and $\check{\mathbf{J}}(e_i) = \check{\mathbf{e}}_i$, where the superposition $\check{\mathbf{J}} \circ \check{\mathbf{J}} = -\mathbf{I}$, for \mathbf{I} being the unity matrix.¹¹

¹⁰It should be noted that on spaces of arbitrary dimension $n + m, m \neq n$, we have $\mathbf{D} = (hD; vD) = \{\Gamma_{\beta\gamma}^\alpha = (L_{jk}^i, L_{bk}^a; C_{jc}^i, C_{bc}^a)\}$; in such cases we can not identify L_{jk}^i with L_{bk}^a and C_{jc}^i with C_{bc}^a . On a tangent bundle TM , the h- and v-indices can be identified in the form " $i \rightarrow a$ ", where $i = 1 = a = n + 1, i = 2 = a = n + 2, \dots, i = n = a = n + n$.

¹¹The formula (13) was obtained in [23, 24] for arbitrary metric compatible affine connections on a manifold or for lifts to tangent bundles. It holds true also for arbitrary metric compatible d-connections. We note that in [23] there is a "-" sign because the authors define there the Nijenhuis tensor with a different sign than in our case, see (14); they also use computations with respect to coordinate bases.

We note that representing the metric in the form (6) we are able to define the almost symplectic structure in a canonical form,

$$\check{\theta} = g_{ij}(x, y) \check{e}^i \wedge e^j, \quad (15)$$

associated to $\check{\mathbf{J}}$ following formulas $\check{\theta}(\mathbf{X}, \mathbf{Y}) \doteq \mathbf{g}(\check{\mathbf{J}}\mathbf{X}, \mathbf{Y})$, which have the same h- and v-components of metric. Defining an almost Kähler structure $(\mathbf{V}^{2n}, \check{\mathbf{J}}, \check{\theta})$ for a N-anholonomic manifold \mathbf{V}^{2n} enabled with the N-connection $\check{\mathbf{N}} = \{\check{N}_j^{a'}\}$, we are able to treat $g_{ij}(x, y)$ as a generalized Lagrange metric but for a N-anholonomic manifold, see discussions in [16, 18].

In Refs. [20, 21], there are considered almost Hermitian models for generalized Lagrange spaces on tangent bundles, when the fiber N-adapted frame structure is holonomic. If we take an arbitrary complex structure J and define the almost symplectic form by the coefficients $\theta_{\alpha\beta} = g_{\alpha\beta}(JX, Y)$, the constructions are not adapted to the N-connection splitting. We have to work with the Levi Civita connection ∇ and perform not N-adapted constructions. In this way we do not have similar constructions as in Lagrange-Finsler geometry (the last ones have the very important property that $g_{ij} = h_{ij}$, for certain canonical N-connection structures) and we cannot apply the N-connection formalism in order to perform a canonical nonholonomic deformation quantification. In this paper, considering a corresponding nonholonomic distribution for the v-subspace, we are able to elaborate constructions with closed symplectic forms resulting in almost Kähler geometry.

The nontrivial N-adapted coefficients of curvature $\hat{\mathbf{R}}_{\beta\gamma\tau}^\alpha$ of $\hat{\mathbf{D}}$ are

$$\begin{aligned} \hat{R}_{hjk}^i &= \mathbf{e}_k \hat{L}_{hj}^i - \mathbf{e}_j \hat{L}_{hk}^i + \hat{L}_{hj}^m \hat{L}_{mk}^i - \hat{L}_{hk}^m \hat{L}_{mj}^i - \hat{C}_{ha}^i \Omega_{kj}^a, \\ \hat{P}_{jka}^i &= e_a \hat{L}_{jk}^i - \hat{\mathbf{D}}_k \hat{C}_{ja}^i, \quad \hat{S}_{bcd}^a = e_d \hat{C}_{bc}^a - e_c \hat{C}_{bd}^a + \hat{C}_{bc}^e \hat{C}_{ed}^a - \hat{C}_{bd}^e \hat{C}_{ec}^a. \end{aligned} \quad (16)$$

For the d-connection ${}^K\mathbf{D}$, with respect to $\check{e}_{\nu'}$ and $\check{e}^{\nu'}$, one has the formulae

$${}^K\mathbf{T}_{\beta'\gamma'}^{\alpha'} = \check{e}_{\alpha'}^{\alpha'} \check{e}_{\beta'}^\beta \check{e}_{\gamma'}^\gamma \hat{\mathbf{T}}_{\beta\gamma}^\alpha \quad \text{and} \quad {}^K\mathbf{R}_{\beta'\gamma'\tau'}^{\alpha'} = \check{e}_{\alpha'}^{\alpha'} \check{e}_{\beta'}^\beta \check{e}_{\gamma'}^\gamma \check{e}_{\tau'}^\tau \hat{\mathbf{R}}_{\beta\gamma\tau}^\alpha, \quad (17)$$

where $\hat{\mathbf{T}}_{\beta\gamma}^\alpha$ and $\hat{\mathbf{R}}_{\beta\gamma\tau}^\alpha$ have nontrivial coefficients given respectively by formulas (11) and (16).

It should be emphasized that we can work equivalently with both connections ∇ and $\hat{\mathbf{D}}$, because they are defined in a unique form for the same metric structure (5). All data computed for one connection can be recomputed for another one by using the distorsion tensor, $\hat{\mathbf{Z}}_{\beta\gamma}^\alpha$, also uniquely defined by the metric tensor \mathbf{g} for the corresponding N-connection splitting, when $\hat{\Gamma}_{\beta\gamma}^\alpha = {}^I\Gamma_{\beta\gamma}^\alpha + \hat{\mathbf{Z}}_{\beta\gamma}^\alpha$.

We conclude that a (semi) Riemannian N-anholonomic manifold provided with a metric \mathbf{g} and an N-connection \mathbf{N} structure can be equivalently described as a usual Riemannian manifold enabled with the connection $\nabla(\mathbf{g})$ (in a form not adapted to the N-connection structure) or as a Riemann–Cartan manifold with the torsion $\hat{\mathbf{T}}(\mathbf{g})$ induced canonically by \mathbf{g} and \mathbf{N} (adapted to the N-connection structure). For an equivalent N-anholonomic structure, we can work with ${}^K\mathbf{T}(\mathbf{g})$, induced canonically by \mathbf{g} and $\check{\mathbf{N}}$ and model the constructions as for almost Kähler spaces.

3 N-anholonomic Fedosov’s Quantization

Since we are interested in providing a natural deformation quantization for (semi) Riemannian spaces equipped with nonholonomic distributions, we have to reformulate the Fedosov approach [7, 8, 9] for N-anholonomic manifolds.

In this section, we perform the geometric constructions of Ref. [23] in a N-adapted form, with respect to $\check{\mathbf{e}}_\nu = (\check{\mathbf{e}}_i, e_{a'})$ and $\check{\mathbf{e}}^\mu = (e^i, \check{\mathbf{e}}^{a'})$, for a d-metric \mathbf{g} (6) and d-connection ${}^K\mathbf{D}$ (12). Proofs of the results will be omitted because they are completely similar to those for Lagrange–Finsler spaces [18], in N-adapted form, and to those for metric affine connections, in coordinate (not N-adapted) form, see [23].

Let us denote by $C^\infty(V)[[v]]$ the space of formal series in the variable v with coefficients from a $C^\infty(V)$ on a Poisson manifold $(V, \{\cdot, \cdot\})$. An associative algebraic structure on $C^\infty(V)[[v]]$, with a v -linear and v -adically continuous star product

$${}^1f * {}^2f = \sum_{r=0}^{\infty} {}_rC({}^1f, {}^2f) v^r, \quad (18)$$

is defined, where ${}_rC, r \geq 0$, are bilinear operators on $C^\infty(V)$, ${}_0C({}^1f, {}^2f) = {}^1f {}^2f$, ${}_1C({}^1f, {}^2f) - {}_1C({}^2f, {}^1f) = i\{{}^1f, {}^2f\}$ and i is the complex unity.

On \mathbf{V}^{2n} enabled with a d-metric structure \mathbf{g} , with respect to $\check{\mathbf{e}}_\nu = (\check{\mathbf{e}}_i, e_{a'})$, we introduce the tensor $\check{\mathbf{A}}^{\alpha\beta} \doteq \check{\theta}^{\alpha\beta} - i \check{\mathbf{g}}^{\alpha\beta}$, where $\check{\theta}^{\alpha\beta}$ is the form (15) with "up" indices and $\check{\mathbf{g}}^{\alpha\beta}$ is the inverse to (g_{ij}, g_{ab}) stated by coefficients of (6). The local coordinates on \mathbf{V}^{2n} are parametrized $u = \{u^\alpha\}$ and the local coordinates on $T_u\mathbf{V}^{2n}$ are labelled $(u, z) = (u^\alpha, z^\beta)$, where z^β are the second order fiber coordinates. We use the formal Wick product

$$a \circ b(z) \doteq \exp\left(i \frac{v}{2} \check{\mathbf{A}}^{\alpha\beta} \frac{\partial^2}{\partial z^\alpha \partial z_{[1]}^\alpha}\right) a(z) b(z_{[1]})|_{z=z_{[1]}}, \quad (19)$$

for two elements a and b defined by formal series of type

$$a(v, z) = \sum_{r \geq 0, |\underbrace{\alpha}_{\alpha}| \geq 0} a_{r, \underbrace{\alpha}_{\alpha}}(u) z^{\underbrace{\alpha}_{\alpha}} v^r, \quad (20)$$

where $\underbrace{\alpha}_{\alpha}$ is a multi-index, defining the formal Wick algebra $\check{\mathbf{W}}_u$, for $u \in \mathbf{V}^{2n}$ associated with the tangent space $T_u \mathbf{V}^{2n}$.

The fibre product (19) is trivially extended to the space of $\check{\mathbf{W}}$ -valued \mathbf{N} -adapted differential forms $\check{\mathcal{W}} \otimes \Lambda$ by means of the usual exterior product of the scalar forms Λ , where $\check{\mathcal{W}}$ denotes the sheaf of smooth sections of $\check{\mathbf{W}}$. For instance, in Ref. [18], we put the left label \mathcal{L} to similar values in order to emphasize that the constructions are adapted to the canonical \mathbf{N} -connection structure induced by a regular effective Lagrangian; in this paper, we do not apply such effective constructions and work with different types of \mathbf{d} -metric, \mathbf{N} -connection and \mathbf{d} -connection structures. All formulae presented below have certain analogies in Lagrange–Finsler geometry (this fact emphasizes the generality of Fedosov’s constructions) but in our case they will define geometric constructions in nonholonomic (semi) Riemannian spaces, or in Einstein spaces, and their almost Kähler models.

We can introduce a standard grading on Λ , denoted \deg_a , and to introduce grading \deg_v, \deg_s, \deg_a on $\mathcal{W} \otimes \Lambda$ defined on homogeneous elements $v, z^\alpha, \check{\mathbf{e}}^\alpha$ as follows: $\deg_v(v) = 1$, $\deg_s(z^\alpha) = 1$, $\deg_a(\check{\mathbf{e}}^\alpha) = 1$, and all other gradings of the elements $v, z^\alpha, \check{\mathbf{e}}^\alpha$ are set to zero. The product \circ from (19) on $\check{\mathcal{W}} \otimes \Lambda$ is bi-graded, we write w.r.t the grading $Deg = 2\deg_v + \deg_s$ and the grading \deg_a .

We extend ${}^K\mathbf{D} = \{ {}^K\Gamma_{\alpha\beta}^\gamma \}$ (12) to an operator on $\check{\mathcal{W}} \otimes \Lambda$,

$${}^K\mathbf{D}(a \otimes \lambda) \doteq \left(\check{\mathbf{e}}_\alpha(a) - u^\beta {}^K\Gamma_{\alpha\beta}^\gamma z^\gamma \check{\mathbf{e}}_\alpha(a) \right) \otimes (\check{\mathbf{e}}^\alpha \wedge \lambda) + a \otimes d\lambda, \quad (21)$$

where ${}^z\check{\mathbf{e}}_\alpha$ is a similar to $\check{\mathbf{e}}_\alpha$ but depending on z -variables. The \mathbf{d} -connection ${}^K\mathbf{D}$ is a \mathbf{N} -adapted \deg_a -graded derivation of the distinguished algebra $(\check{\mathcal{W}} \otimes \Lambda, \circ)$, in brief, one call \mathbf{d} -algebra (one follows from (19) and (21)).

The Fedosov operators $\check{\delta}$ and $\check{\delta}^{-1}$ are defined, in our case, on $\check{\mathcal{W}} \otimes \Lambda$,

$$\check{\delta}(a) = \check{\mathbf{e}}^\alpha \wedge {}^z\check{\mathbf{e}}_\alpha(a) \text{ and } \check{\delta}^{-1}(a) = \begin{cases} \frac{i}{p+q} z^\alpha \check{\mathbf{e}}_\alpha(a), & \text{if } p+q > 0, \\ 0, & \text{if } p=q=0, \end{cases}$$

where $a \in \check{\mathcal{W}} \otimes \Lambda$ is homogeneous w.r.t. the grading \deg_s and \deg_a with $\deg_s(a) = p$ and $\deg_a(a) = q$. Such operators define the formula $a =$

$(\check{\delta} \check{\delta}^{-1} + \check{\delta}^{-1} \check{\delta} + \sigma)(a)$, where $a \mapsto \sigma(a)$ is the projection on the (\deg_s, \deg_a) -bihomogeneous part of a of degree zero, $\deg_s(a) = \deg_a(a) = 0$. We note that $\check{\delta}$ is also a \deg_a -graded derivation of the d -algebra $(\check{\mathcal{W}} \otimes \mathbf{\Lambda}, \circ)$.

Using the extension of ${}^K\mathbf{D}$ to $\check{\mathcal{W}} \otimes \mathbf{\Lambda}$, we construct the operators

$${}^K\mathcal{T} \doteq \frac{z^\gamma}{2} \check{\theta}_{\gamma\tau} {}^K\mathbf{T}_{\alpha\beta}^\tau(u) \check{\mathbf{e}}^\alpha \wedge \check{\mathbf{e}}^\beta, \quad (22)$$

$${}^K\mathcal{R} \doteq \frac{z^\gamma z^\varphi}{4} \check{\theta}_{\gamma\tau} {}^K\mathbf{R}_{\varphi\alpha\beta}^\tau(u) \check{\mathbf{e}}^\alpha \wedge \check{\mathbf{e}}^\beta, \quad (23)$$

where the nontrivial coefficients of ${}^K\mathbf{T}_{\alpha\beta}^\tau$ and ${}^K\mathbf{R}_{\varphi\alpha\beta}^\tau$ are computed as in (17). One has the important formulae:

$$[{}^K\mathbf{D}, \check{\delta}] = \frac{i}{v} ad_{Wick}({}^K\mathcal{T}) \text{ and } {}^K\mathbf{D}^2 = -\frac{i}{v} ad_{Wick}({}^K\mathcal{R}),$$

where $[\cdot, \cdot]$ is the \deg_a -graded commutator of endomorphisms of $\check{\mathcal{W}} \otimes \mathbf{\Lambda}$ and ad_{Wick} is defined via the \deg_a -graded commutator in $(\check{\mathcal{W}} \otimes \mathbf{\Lambda}, \circ)$.

We reformulate three theorems¹² and some fundamental properties of Fedosov's d -operators for N -anholonomic (semi) Riemannian and Einstein spaces:

Theorem 3.1 *Any d -tensor (6) defines a flat canonical Fedosov d -connection*

$${}^K\mathcal{D} \doteq -\check{\delta} + {}^K\mathbf{D} - \frac{i}{v} ad_{Wick}(r)$$

satisfying the condition ${}^K\mathcal{D}^2 = 0$, where the unique element $r \in \check{\mathcal{W}} \otimes \mathbf{\Lambda}$, $\deg_a(r) = 1$, $\check{\delta}^{-1}r = 0$, solves the equation

$$\check{\delta}r = {}^K\mathcal{T} + {}^K\mathcal{R} + {}^K\mathbf{D}r - \frac{i}{v} r \circ r$$

and this element can be computed recursively with respect to the total degree Deg as follows:

$$\begin{aligned} r^{(0)} &= r^{(1)} = 0, \quad r^{(2)} = \check{\delta}^{-1} {}^K\mathcal{T}, \\ r^{(3)} &= \check{\delta}^{-1} \left({}^K\mathcal{R} + {}^K\mathbf{D}r^{(2)} - \frac{i}{v} r^{(2)} \circ r^{(2)} \right), \\ r^{(k+3)} &= \check{\delta}^{-1} \left({}^K\mathbf{D}r^{(k+2)} - \frac{i}{v} \sum_{l=0}^k r^{(l+2)} \circ r^{(l+2)} \right), \quad k \geq 1. \end{aligned}$$

¹²see Theorems 4.1, 4.2 and 4.3 in [17], for Lagrange-Finsler spaces; for effective Lagrange spaces, see [18]; all such theorems are N -adapted generalizations of the original Fedosov's results

where we denoted the Deg-homogeneous component of degree k of an element $a \in \check{\mathcal{W}} \otimes \mathbf{\Lambda}$ by $a^{(k)}$.

Theorem 3.2 *A star-product on the almost Kähler model of a N-anholonomic (pseudo) Riemannian space is defined on $C^\infty(\mathbf{V}^{2n})[[v]]$ by formula*

$${}^1f * {}^2f \doteq \sigma(\tau({}^1f)) \circ \sigma(\tau({}^2f)),$$

where the projection $\sigma : \check{\mathcal{W}}_{K\mathcal{D}} \rightarrow C^\infty(\mathbf{V}^{2n})[[v]]$ onto the part of \deg_s -degree zero is a bijection and the inverse map $\tau : C^\infty(\mathbf{V}^{2n})[[v]] \rightarrow \check{\mathcal{W}}_{K\mathcal{D}}$ can be calculated recursively w.r.t the total degree Deg,

$$\begin{aligned} \tau(f)^{(0)} &= f, \\ \tau(f)^{(k+1)} &= \check{\delta}^{-1} \left({}^K\mathbf{D}\tau(f)^{(k)} - \frac{i}{v} \sum_{l=0}^k ad_{Wick}(r^{(l+2)})(\tau(f)^{(k-l)}) \right), \end{aligned}$$

for $k \geq 0$.

Let ${}^f\xi$ be the Hamiltonian vector field corresponding to a function $f \in C^\infty(\mathbf{V}^{2n})$ on the space $(\mathbf{V}^{2n}, \check{\theta})$ and consider the antisymmetric part

$${}^-C({}^1f, {}^2f) \doteq \frac{1}{2} (C({}^1f, {}^2f) - C({}^2f, {}^1f))$$

of a bilinear operator $C({}^1f, {}^2f)$. We say that a star-product (18) is normalized if ${}_1C({}^1f, {}^2f) = \frac{i}{2} \{ {}^1f, {}^2f \}$, where $\{ \cdot, \cdot \}$ is the Poisson bracket. For a normalized $*$, the bilinear operator ${}_2C$ is a de Rham-Chevalley 2-cocycle. There is a unique closed 2-form $\check{\varkappa}$ such that

$${}_2C({}^1f, {}^2f) = \frac{1}{2} \check{\varkappa}({}^{f_1}\xi, {}^{f_2}\xi) \quad (24)$$

for all ${}^1f, {}^2f \in C^\infty(\mathbf{V}^{2n})$. The class c_0 of a normalized star-product $*$ is defined as the equivalence class $c_0(*) \doteq [\check{\varkappa}]$.

A straightforward computation of ${}_2C$ from (24) and the results of Theorem 3.1 give the proof of

Lemma 3.1 *The unique 2-form can be computed*

$$\begin{aligned} \check{\varkappa} &= -\frac{i}{8} \check{\mathbf{J}}_\tau^{\alpha'} {}^K\mathbf{R}_{\alpha'\alpha\beta}^\tau \check{\mathbf{e}}^\alpha \wedge \check{\mathbf{e}}^\beta - i \check{\lambda}, \\ \check{\lambda} &= d\check{\mu}, \quad \check{\mu} = \frac{1}{6} \check{\mathbf{J}}_\tau^{\alpha'} {}^K\mathbf{T}_{\alpha'\beta}^\tau \check{\mathbf{e}}^\beta. \end{aligned}$$

Let us define the canonical class $\check{\varepsilon}$ for ${}^{\check{N}}T\mathbf{V}^{2n} = h\mathbf{V}^{2n} \oplus v\mathbf{V}^{2n}$ with the left label related to a N-connection structure $\check{\mathbf{N}}$. The distinguished complexification of such second order tangent bundles can be performed in the form $T_{\mathbb{C}}({}^{\check{N}}T\mathbf{V}^{2n}) = T_{\mathbb{C}}(h\mathbf{V}^{2n}) \oplus T_{\mathbb{C}}(v\mathbf{V}^{2n})$. In this case, the class $\check{\varepsilon}$ is the first Chern class of the distributions $T'_{\mathbb{C}}({}^{\check{N}}T\mathbf{V}^{2n}) = T'_{\mathbb{C}}(h\mathbf{V}^{2n}) \oplus T'_{\mathbb{C}}(v\mathbf{V}^{2n})$ of couples of vectors of type $(1, 0)$ both for the h- and v-parts.

We calculate the canonical class $\check{\varepsilon}$, using the d-connection ${}^K\mathbf{D}$ considered for constructing $*$ and the h- and v-projections $h\Pi = \frac{1}{2}(Id_h - iJ_h)$ and $v\Pi = \frac{1}{2}(Id_v - iJ_v)$, where Id_h and Id_v are respective identity operators and J_h and J_v are almost complex operators, which are projection operators onto corresponding $(1, 0)$ -subspaces. Defining the matrix $(h\Pi, v\Pi) {}^K\mathbf{R} (h\Pi, v\Pi)^T$, where $(\dots)^T$ means transposition, which is the curvature matrix of the N-adapted restriction of ${}^K\mathbf{D}$ to $T'_{\mathbb{C}}({}^{\check{N}}T\mathbf{V}^{2n})$, we compute the closed Chern-Weyl form

$$\begin{aligned} \check{\gamma} &= -iTr \left[(h\Pi, v\Pi) {}^K\mathbf{R} (h\Pi, v\Pi)^T \right] = -iTr \left[(h\Pi, v\Pi) {}^K\mathbf{R} \right] \\ &= -\frac{1}{4} \check{\mathbf{J}}_{\tau}^{\alpha'} {}^K\mathbf{R}^{\tau}_{\alpha'\alpha\beta} \check{\mathbf{e}}^{\alpha} \wedge \check{\mathbf{e}}^{\beta}. \end{aligned}$$

We get that the canonical class is $\check{\varepsilon} \doteq [\check{\gamma}]$.

Theorem 3.3 *The zero-degree cohomology coefficient $c_0(*)$ for the almost Kähler model of a (pseudo) Riemannian space defined by d-tensor (6) is given by*

$$c_0(*) = -(1/2i) \check{\varepsilon}.$$

For a particular case when (6) is a solution of the Einstein equations, the coefficient $c_0(*)$ defines certain quantum properties of the gravitational field. Any metric defining a classical Einstein manifold can be nonholonomically deformed into the corresponding quantum configuration.

Finally, in this section we discuss why we have not considered directly the nonholonomic quantum deformations of Einstein's equations. In general this is an important subject, for further detailed considerations see [25] on explicit constructions with well defined limits, for instance, in loop gravity and/or any string/ gauge models of gravity. In Refs. [26, 27], we reformulated/ extended the Einstein equations as Yang-Mills equations for the Cartan connection in affine / de Sitter frame bundles and considered non-commutative generalizations of general relativity via Witten-Seiberg maps. Similar constructions can be elaborated in the nonholonomic deformation

quantization methods of the above-mentioned models. Even gauge theories with affine/ Poincare gauge groups are endowed with degenerate Killing forms and have an undefined Lagrange structure in the total space. Such theories can be quantized following respective methods of geometric and/or BRST quantization [28, 29].

4 Conclusion

In this paper we have provided a natural Fedosov deformation quantization of (semi) Riemannian spaces equipped with nonholonomic distributions when the spacetime geometry is modelled equivalently on almost Kähler manifolds. This approach was elaborated following a synthesis of the nonlinear connection (N-connection) formalism in Lagrange–Finsler geometry and certain deformation quantization methods. It also allows to quantize any solution of Einstein’s equations.

In Ref. [18] we developed a deformation quantization scheme for general relativity by extending, canonically, the geometric structures in extra dimensions or in tangent bundles of spacetime manifolds. That approach, in general, violates the local Lorentz symmetry which has captured certain interest in the current literature. In this paper the nonholonomic deformations are considered for the same spacetime manifold which is being quantized by geometric methods. This allows us to preserve a formal local Lorentz invariance even when one works with canonical distinguished connections which are metric compatible and with a nonholonomically induced torsion. All non-quantized and quantized expressions can be redefined for the Levi Civita connection and this is possible because all linear connections considered in our approach are induced uniquely by the metric structure.

It should be emphasized that by prescribing a nonholonomic distribution, which defines a nonholonomic frame structure with an associated nonlinear connection on a (semi) Riemannian manifold, we are not violating the general covariance principle. For certain purposes we only constrain some frame components to be nonintegrable distributions, but this does not affect the fundamental properties of physical interactions. In general, all constructions can be re-defined for arbitrary frame and coordinate systems.

Any $2+2$ splitting of an Einstein manifold can be performed in a natural nonholonomic fashion in order to generate the required almost symplectic structures and apply, straightforwardly, the deformation quantization methods. In this way we deform both the frame and linear connections structures in a canonical fashion when the constructions are defined in terms of the

metric structure (in particular, by a solution of the Einstein equations).

The study of nonholonomic geometry and deformation quantization of gravity in connection to Loop Quantum Gravity, canonical and perturbative approaches, noncommutative generalizations and applications to modern cosmology and gravitational physics will be the subject of future investigation.

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